Abstract—In this paper, the filtering problem for linear Gaussian system is considered. We will propose a self-tuning asynchronous filter subject to measurement-lossy situation where the measurements are available at equal sampling intervals, namely, the process and measurement equations are asynchronous at every time but synchronous every several times. What is more, considering the variation of noise statistics, a regular called noise variance estimator is introduced to adjust filter’s coefficients adaptively. The case studies of wheeled robot navigation system and air quality evaluation system will show the practicability in engineering and some characteristics.

Index Terms—linear Gaussian system, lossy measurement, wheeled robot navigation system, air quality evaluation system

I. INTRODUCTION

The filtering problem, which concentrates on how to filter the stochastic disturbances and measurement error or reduce the affect brought by them, has been a hot topic in the domain of control theory and signal processing since the Wiener Filter come out. Their applications can be found in almost every domains. Up to now, there has been developed various filters such as Kalman filter, Extend Kalman filter, Bayesian filter, particle filter and those improved filters based on them. However, it still has a lot of problems to be solved, e.g. the compromise between the filtering accuracy and computational complexity. This paper will concentrate on the filtering under the measurement and process are not synchronous.

The linear Gaussian system has been proven to be an imprecise model describing the real-world object in most situations for it is only a approximation with respect to the nonlinear non-Gaussian system. However, it is still a preferable modelling to solve engineering problems due to its conciseness which may facilitate the solution. As to the filtering problem, the Kalman filter is an optimal estimation for linear Gaussian system [1], but it still exists a lot of practice problems to be solved. There has emerged some improved Klampan filter to overcome the drawbacks of standard Kalman filter. For the time-invariant singular system mixed with multiplicative noises where the delayed measurements exists, the improved Kalman filter is proposed by utilizing standard singular value decomposition [2]. Considering the system with high-frequency correlated noises whose variances are not completely known, the wavelet-adaptive Kalman filter is proposed in that the wavelet technology has higher frequency resolution and lower time resolution in the low frequency region and the opposite performance in the high frequency region [3]. The presence of large model uncertainty will affect the filtering performance a lot and to obtain a precise model by thousands of experiments is not cost-effective, the MVDP-FMKF is developed under the assumption that the model uncertainty can be equally replaced by a finite set of different known models [4]. In the case whose measurements noises are censored Gaussian or correlated to the process states, the named Tobit Kalman filter is introduced as an estimator for censored measurements [5]. In conclusion, there are a variety of improved filters been developed on the standard Kalman filter, thus demonstrating its powerful vitality.

For most of the filters are corrected by measurements and the delays or drops occur very often such that many researcher have done a lot of work on the filter design under the situation measurements are lossy or delayed. For linear discrete stochastic systems with multiple packet dropouts, packet dropout model and its optimal linear estimation problem is proposed [6]. Without the assistant of state augmentation, the system is transformed to that a moving average colored measurement noise [7]. Considering all these three measurement uncertainties of random measurement delays, packet dropouts and missing measurements, which have certain probability of occurrence in the network, the adaptive filter are proposed [8]. For the networked system with lossy measurements which may raise during the communication over wireless channels, the Markovian jumping linear system is introduced to model this measurement-lossy process [9]. There still have a lot of other filters been proposed in different situations, e.g. one-step random delays and so on [10]–[12]. In this paper, we will propose a self-tuning filter subject to measurement-lossy situation where the measurements are available in equal sampling interval, namely, the process and measurement equations are not synchronous. What is more, considering the variation of noise statistics, a regular called noise variance estimator is introduced to adjust filter’s coefficients adaptively.

The proposed algorithm has a typical application in wheeled robot navigation system. The wheeled robot navigation system, determining its position and attitude, plays the primarily
important role which has significant influence to the its performance on account of the current and future decisions are made based on the navigation data. For the lack of precision for long period, the autonomous navigation usually dose not work independently. The external references are used to conduct the correction by the other means, including the GPS or the computer vision which is quite popular these years [13]–[15]. However, this type of systems are suffering to one problem that is the correction can not be synchronous with the autonomous navigation for the means of GPS or computer vision are time-consuming thus causing a longer sampling interval. Hence, the proposed algorithm can be used to solve this problem. Another application of the proposed algorithm can be found in the atmospheric dispersion system to estimate the street air pollutant concentration in real time, which is of great significance due to its dominance for government to make more effective control strategies to improve our atmospheric environment. The conventional method be referred as model-based method is to employ the auto regression (AR) model or artificial neural network (ANN) model to describe the relationship of the pollutant concentration by other variables such as emission rate, vehicle characteristics, traffic and meteorology conditions [16], [17]. The alternative is called mobile-sensing method which detects the pollutant concentration directly with a monitoring car equipped with air quality sensors and positioning device which can draw out a temporal- and spatial-distribution map for pollutant concentration while driving around the city [18]. To summarize these two methods it can be found that the model-based method performs in a precise and fast way in a short period for its required data can be obtained by the existing sensors distributed on the road network, but the accuracy will deteriorate over time for the error accumulation. On the other hand, the mobile-sensing method is able to realize the precise measurement for a long time but not in real time in that every street can be detected only once within one non-overlapping patrol. That is to say, the model-based method and mobile-sensing method can not be synchronized which makes it possible to apply the proposed algorithm to combine the two method based on their complementary characteristics.

The rest of the paper is organized as follows. The Section 2 will formulate the problem mathematically and Section 3 for the solution to this problem in detail. The section 4 will contain two simulation studies of two cases, the application of the proposed algorithm in wheeled robot navigation system and air quality evaluation system which are mentioned before. The paper is concluded in the lase section.

II. PROBLEM STATEMENT

In this section it will be presented the problem to be solved in mathematical form. Considering the linear time-invariant discrete system of interest described by the linear stochastic difference equation as

\[ x_{t+1} = Fx_t + Gu_t + \Gamma w_t \tag{1a} \]

\[ y_{nt} = Hx_{nt} + v_{nt} \tag{1b} \]

where \( F, G, \Gamma \) and \( H \) are known time-invariant matrices with proper dimensions, \( t = 0, 1, \cdots \) respects for discrete time which has naturally dropped the time internal parameter, \( x_t \in \mathbb{R}^n \) respects for states, \( y_t \in \mathbb{R}^m \) for measurements, \( u_t \in \mathbb{R}^r \) for controls, \( w_t \in \mathbb{R}^n \) and \( v_t \in \mathbb{R}^m \) for process- and measurement-noises, respectively. Some assumptions for this system is listed as follows.

\textbf{Assumption 1}: The noises are independent Gaussian white processes that

\[ \mathbb{E}\{w_t \} = \mathbb{E}\{v_t \} = 0 \]

\[ \mathbb{E}\{w_t w_t^T\} = \mathbb{E}\{v_t v_t^T\} = Q_t \]

\[ \mathbb{E}\{w_t v_t^T\} = 0 \]

where \( Q \) and \( R \) are variance matrices of \( w_t \) and \( v_t \), respectively; \( \delta_t \) is the Kronecker function, the symbol \( \mathbb{E} \) represents for mathematical expectation and prime for the transposition.

\textbf{Assumption 2}: The pairwise structure \((\mathcal{F}, H)\) are completely observable, namely, the observability matrix \( O = [H, H\mathcal{F}, \cdots, H\mathcal{F}^{N-1}] \) are full-rank where \( \mathcal{F} = F^N \).

\textbf{Assumption 3}: The noise variance matrices \( Q \) and \( R \) are unknown or slowly changing.

Therefore the problem is to develop an linear minimum variance estimator \( \hat{x}_t \) based on the measurement set \( \{y_{t0}, y_{t1}, \cdots, y_{t[N]}\} \) where \( [t] \) is the largest integer not larger than \( t \).

III. SELF-TUNING ASYNCHRONOUS FILTERING

The proposed algorithm will be developed on the standard Kalman Filter, namely, an improved Kalman Filter. A noise variance estimator will be introduced to eliminate the affect brought by noise uncertainty.

A. Improved Kalman Filter

Firstly the brief review of the estimation procedure of standard Kalman Filter will be presented first to inspire the development. The Kalman Filter is a set of mathematical equations providing a estimation algorithm in the recursive form to minimize the squared error. It has been proved to be an optimal estimator for linear Gaussian systems thus to be frequently employed in engineering, e.g. the multi-sensor-fusion problem. What is more, it estimates the process states and then obtains feedback in the form of measurement, dividing the equations into two groups: predictor equation and corrector equation. The former is responsible for projecting forward the current estimations of state and error covariance to obtain the a priori estimations for the next time step. The latter is responsible for the feedback, that is, incorporating a new measurement into the a priori estimations to obtain an improved a posteriori estimation. The process is repeated with the previous a posteriori estimation used to predict the new a priori estimation. Hence, the Kalman Filter performs as a time-variant recursive optimal filter.

Suppose that \( \{y_{nt}, t = 0, 1, \cdots\} \) is obtained at time \( Nt \), coupled with the optimal estimation \( \hat{x}_{nt} \) for \( x_{nt} \) which is the linear summation of \( \{y_{nt}, t = 0, 1, \cdots\} \). For the next time without measurement, we have

\[ \hat{x}_{nt+1} = F\hat{x}_{nt} + Gu_{nt} \tag{3} \]
which is obviously an unbiased estimation. Defining the estimation error \( \bar{x}_t = x_t - \hat{x}_t \) and its variance \( P_t = \mathbb{E}[\bar{x}_t \bar{x}_t'] \) yields

\[
P_{Nt+1} = \mathbb{E}\{(\bar{x}_{Nt+1}, \bar{x}_{Nt+1}')\} = \mathbb{E}\{(\hat{x}_{Nt+1} - \bar{x}_{Nt+1}, \bar{x}_{Nt+1} - \bar{x}_{Nt+1}')\} = \mathbb{E}\{F\hat{x}_N, F' + \Gamma \sigma_t \sigma_t'\} = FP_{Nt} F' + \Gamma \sigma_t \sigma_t'
\]

which is essentially the predicted estimation error variance in the standard Kalman Filter.

For the absence of measurements it is equal to the corrected one. Hence, during the period of \( \{Nt+\tau, \tau = 1, 2, \cdots, N-1\} \) we have

\[
\tilde{x}_{Nt+\tau} = F \tilde{x}_{Nt+\tau-1} + Gu_{Nt+\tau-1}
\]

with the estimation error variance of

\[
P_{Nt+\tau} = FP_{Nt+\tau-1} F' + \Gamma \sigma_t \sigma_t'
\]

At time \( Nt+N \) with the presence of measurement, we have

\[
\tilde{x}_{Nt+N} = \mathcal{F} \tilde{x}_{Nt} + \mu_{Nt}
\]

where \( \mathcal{F} = F^N, \mu_{Nt} = \sum_{i=0}^{N-1} F^{N-i-1} Gu_{Nt+i} \) and \( \sigma_{Nt} = \sum_{i=0}^{N-1} (F^i \sigma_t \sigma_t') F'^i \). The \( \tilde{x}_{Nt+N} \) and \( \tilde{P}_{Nt+N} \) are essentially the predicted state estimation and predicted error variance in a standard Kalman Filter.

Because the correction process is totally similar to the standard Kalman Filter [19], the mathematical deduction will not be covered again. We named the filter as Asynchronous Kalman Filter (ASYNCKF) and its whole procedure is presented in Algorithm 1.

The ASYNCKF algorithm is essentially a time-variant recursive filter by calculating the optimal Kalman Gain \( K_t \) at \( t = 0, N, 2N, \cdots \) which brings large computational complexity especially in the case there are high-dimensional matrices. However, if the \( K_t \) is converged to a constant matrix \( K \) at \( t \to \infty \), the time-invariant filter can be obtained via substituting \( K_t \) by \( K \). This is only available if the process equation (1a) and measurement model (1b) are synchronized.

Considering the period without measurements of \( \{Nt + \tau, \tau = 1, 2, \cdots, N-1\} \) we have

\[
x_{Nt+\tau} = Fx_{Nt+\tau-1} + Gu_{Nt+\tau-1} + \Gamma w_{Nt+\tau-1}
\]

Setting \( \tau = N \) the synchronization model of (1a) and (1b) will be derived as

\[
x_{Nt+N} = Fx_{Nt} + \mu_{Nt} + \omega_{Nt}
\]

\[
y_{Nt} = Hx_{Nt} + v_{Nt}
\]

\[
x_{Nt+N} = \mathcal{F}x_{Nt} + \mu_{Nt} + \omega_{Nt}
\]

\[
y_{Nt} = Hx_{Nt} + v_{Nt}
\]

Algorithm 1: Asynchronous Kalman Filter (ASYNCKF)

**Input**: Measurement set \( \{y_{0}, y_{N}, y_{2N}, \cdots, y_{(N-1)N}\} \), prior statistics \( \chi, \rho, Q \) and \( R \).

**Output**: Estimation set \( \{\tilde{x}_{0}, \cdots, \tilde{x}_{Nt}\} \).

1. **Initialization**:
   - Set \( t := 0, \tilde{x}_0 := \chi \) and \( P_0 := \rho \).
2. **Prediction**:
   - Increase \( t \) by 1.
   - Predicted State Estimation: \( \tilde{x}_t := \tilde{x}_{t-1} + Gu_{t-1} \).
   - Predicted Error Variance: \( P_t := FP_{t-1} F' + \Gamma \sigma_t \sigma_t' \).
3. **Correction**:
   - Optimal Kalman Gain:
     \[ K_t := \left\{ \begin{array}{ll} 0_{\chi \times \rho} & \text{loss measurement} \\ P_{t-1} H' (H P_{t-1} H' + R)^{-1} & \text{others} \end{array} \right. 
     
   - Corrected State Estimation: \( \tilde{x}_t := \tilde{x}_t + K_t(y_t - H \tilde{x}_t) \).
   - Corrected Error Variance: \( P_t := (I - KH)P_t \).
4. **Feedback**:
   - Return to Step 2.

where \( \mathcal{F} = F^N, \mu_{Nt} = \sum_{i=0}^{N-1} F^{N-i-1} Gu_{Nt+i} \) and \( \omega_{Nt} = \sum_{i=0}^{N-1} (F^i \sigma_t \sigma_t') F'^i \) represents for Gaussian white noise with the variance matrix of \( Q = \sum_{i=0}^{N-1} (F^i \sigma_t \sigma_t') F'^i \). The sufficient condition of convergence of \( K_t \) at \( t = 0, N, 2N, \cdots \) is that the controllability matrix \( C = [I, F, F^2, \cdots, F^{N-1}] \) and observability matrix \( O = [H, H F, \cdots, H F^{N-1}] \) are full-rank. The first condition holds obviously and the second one is under the Assumption 2. Then we have the time-invariant form of ASYNCKF as

\[
\tilde{x}_{t+1} = (I - KH)(F \tilde{x}_t + Gu_t) + Ky_t.
\]

where the matrix \( K \) is set as a null matrix at \( t \neq 0, N, 2N, \cdots \) and a non-zero matrix at \( t = 0, N, 2N, \cdots \) whose value can be determined by solving the follow equation set

\[
E = FP' F + Q
\]

\[
K = EH' (HEH' + R)^{-1}
\]

\[
P = (I - KH)F
\]

It has been proved that the time-invariant filter will converge to the optimal time-variant Kalman filter with probability one such that the convergence of the time-invariant form of ASYNCKF can be easily deduced [20]. Specially noted that we use the word time-invariant here to show the invariance of \( K_{Nt} \), the Kalman Gain is switched between two constant matrices instead of an infinite set. This has reduced the computational complexity to a large extent, so the time-invariant form is more applicable in practice in spite of its sub-optimality.

**B. Noise Variance Estimator**

Realizing that the noise statistics are completely or partly unknown, or varying with time. So a regulator will be developed to adjust the parameter adaptively. For it is designed on the measurements, the regulator can be described as noise variance estimator \( \mathcal{Q}_{Nt} \) and \( \bar{R}_{Nt} \), which is the linear function over \( \{y_{Nt}, t = 0, 1, 2, \cdots\} \). Rewriting (9a) yields

\[
x_{Nt+N} = q^{-N} \mathcal{F}x_{Nt} + G \mu_{Nt} + \Gamma \omega_{Nt}
\]
where $q^{-1}$ is the backward shift operator that $q^{-1}x_{i+1} = x_i$, and then yields

$$x_{Nt+N} = (I - q^{-N}\mathcal{F})(G\mu_{Nt} + \Gamma\omega_{Nt})$$  \hspace{1cm} (13)

Substituting (13) into (9b) yields

$$y_{Nt} = H(I - q^{-N}\mathcal{F})(Gq^{-N}\mu_{Nt} + \Gamma q^{-N}\omega_{Nt}) + v_{Nt}$$  \hspace{1cm} (14)

and then using the equation

$$I - q^{-N}\mathcal{F} = \frac{\text{adj}(I - q^{-N}\mathcal{F})}{\text{det}(I - q^{-N}\mathcal{F})}$$  \hspace{1cm} (15)

where $\text{adj}$ represents for adjacent matrix and $\text{det}$ for determinant. The (14) becomes

$$\text{det}(I - q^{-N}\mathcal{F})y_{Nt} = H\text{adj}(I - q^{-N}\mathcal{F})(Gq^{-N}\mu_{Nt} + \Gamma q^{-N}\omega_{Nt}) + \text{det}(I - q^{-N}\mathcal{F})v_{Nt}$$  \hspace{1cm} (16)

and then conducting the left-coprime factorization [21] on both sides yields

$$A(q^{-N})y_{Nt} = B_\mu(q^{-N})\mu_{Nt} + B_\omega(q^{-N})\mu_{Nt} + A(q^{-N})v_{Nt}$$  \hspace{1cm} (17)

where $A(q^{-N})$, $B_\mu(q^{-N})$ and $B_\omega(q^{-N})$ are polynomial matrices with the form of

$$X(q^{-N}) = X_0 + q^{-N}X_1 + \cdots + q^{-Nt}\mu_{Nt} + \cdots$$

with $A_0 = I$, $B_\mu = 0$. Introducing $z_{Nt} = A(q^{-N})y_{Nt} - B_\mu(q^{-N})\mu_{Nt}$ yields

$$z_{Nt} = B_\omega(q^{-N})\mu_{Nt} + A(q^{-N})v_{Nt}$$  \hspace{1cm} (18)

where $z_{Nt}$ is the summation of two independent moving average processes, thus $z_{Nt}$ being another moving average process whose order is the same as that of the component process of higher order [22].

Defining the correlation function of $z_{Nt}$ as $A^{(i)} = \mathbb{E}(z_{Nt}z_{Nt-1}^{(i)})$ where $i = 0, 1, 2, \cdots, \ell_z$, and $\ell_z = \max(i, A^{(i)} = 0)$. Calculating correction function of (18) yields

$$A^{(i)} = \sum_{k=0}^{\ell_z} (B_{\omega,k}Q B_{\omega,k-1} + A_k R A_{k-1}^T)$$  \hspace{1cm} (19)

where $\ell_z = \max(\ell_x, \ell_y, \ell_z)$, $\ell_z \geq \ell_x$ and $A_k = 0$ at $k > \ell_x$ and $B_{\omega,k} = 0$ at $k > \ell_x$. Expanding (19) by the matrix elements yields the following linear equations as

$$\Delta \theta = \vartheta$$  \hspace{1cm} (20)

where $\theta$ represents for a $\ell_x \times 1$ column vector consisting all the unknown elements in $Q$ and $R$. $\theta$ for a $\ell_y \times 1$ column vector whose elements each consist of a constant plus one element of $A^{(i)}$, $\Delta$ for a known $\ell_x \times \ell_y$ constant matrix. If $\Delta$ has the full column rank, then it has the same row rank equaling $\ell_y$. Selecting $\ell_y$ linear independent equations yields

$$\Delta \theta = \vartheta$$  \hspace{1cm} (21)

where $\Delta$ is known $\ell_x \times \ell_y$ nonsingular constant matrix. So $\theta$ can be solved as

$$\theta = \Delta^{-1}\vartheta$$  \hspace{1cm} (22)

and then subsisting the estimations $\hat{Q}_{Nt}$ and $\hat{R}_{Nt}$ into (22) yields

$$\hat{\theta}_{Nt} = \Delta^{-1}\hat{\vartheta}_{Nt}$$  \hspace{1cm} (23)

where $\hat{\theta}_{Nt}$ contains the estimation $\hat{Q}_{Nt}$ and $\hat{R}_{Nt}$.

So the remaining work is to design the estimators for $A^{(i)}$, that is, the $\hat{A}^{(i)}_{Nt}$. Furthermore considering the variation of noise statistics, the estimator $\hat{A}^{(i)}_{Nt}$ must forget the history data gradually, and thus

$$\hat{A}^{(i)}_{Nt} = \frac{1}{t} \sum_{k=1}^{t} (1 - \gamma^{t-k}) z_{Nt} z_{Nt-1}^T$$  \hspace{1cm} (24a)

$$= \frac{1}{t} \left( \frac{\gamma}{1 - \gamma} \sum_{k=1}^{t-1} \gamma^{t-k} z_{Nt} z_{Nt-1}^T + z_{Nt} z_{Nt-1}^T \right)$$

$$= \frac{1 - \gamma}{\gamma - 1} \hat{A}^{(i)}_{Nt-1} + \frac{1}{t} (z_{Nt} z_{Nt-1}^T - \gamma \hat{A}^{(i)}_{Nt-1})$$  \hspace{1cm} (24b)

where $0 < \gamma < 1$ is said to be the forgetting factor that the forgetting rate growing with the positive number $\gamma$ getting smaller. This is only available at $Nt, t \geq i$ while the estimation at other period can be determined by experience or arbitrarily for its asymptotical optimality. The $\gamma$ should be selected as a larger number for slowly changing noises. To facilitate the implementation in computation, thus giving the recursive form. According to the ergodicity of stationary stochastic process we have $\hat{A}^{(i)}_{Nt} \rightarrow A^{(i)}$ with probability one at $t \rightarrow \infty$. The method in (24a) and (24b) is said to be the Fading Estimation. And the alternative is the Finite-Memory Estimation written as

$$\hat{A}^{(i)}_{Nt} = \frac{1}{\ell_x + 1} \sum_{k=t-\ell_x+1}^{t} z_{Nt+k} z_{Nt+k-1}^T$$  \hspace{1cm} (25)

where the recursive form dose not exist. However, this method is still a promising solution. As seen in Fig. 1, the blue curve represents for the weights assignment on history data of fading estimation, it is found that all the history data are be shrunken sharply except the current ones if the $\gamma$ is set to be a relatively small positive. The forgetting factor is able to control the steepness of the blue curve. As to the finite-memory estimation shown as the red curve performs much better than that of the fading estimation in spite of the inexistence of recursive form. Hence, the fading estimation is preferable in the real-time situation and the finite-memory estimation in the scenario requiring more accuracy.

Subsisting the $\hat{A}^{(i)}_{Nt}$ into (23), the noise variance estimators $\hat{Q}_{Nt}$ and $\hat{R}_{Nt}$ can be then worked out and applied to adjust the Kalman Gain. Specially noted that there is no need to estimate $Q$ in that the correction of $\hat{P}_{Nt}$ can be solved by applying (7b) which involves $Q$ only. There are no doubts that the
noise variance estimator will converge to the real value with probability one, and thus the improved Kalman Filter possesses the optimality. The Algorithm 2 presents the whole procedure as follows.

**Algorithm 2 Noisy Variance Estimator (VAREST)**

**Input:** Measurement set \{y_{N,t}, t = 0, 1, 2, \ldots \}

**Output:** Noise variance estimation set \{\hat{Q}_{N,t}, t = 0, 1, 2, \ldots \} and \{\hat{R}_{N,t}, t = 0, 1, 2, \ldots \}

1. Set \( t := 0, \hat{A}^{(0)}_{N,t} := 0, \hat{A}^{(1)}_{N,t} := 0 \).
2. Set \( 0 < \gamma < 1 \) as an appropriate real number.
3. Increase \( t \) by 1.
4. Calculate correlation by fading estimation
   \[
   \hat{A}^{(0)}_{N,t} := \gamma \hat{A}^{(0)}_{N,t-1} + \frac{1}{\gamma} (z_{N,t} z'_{N,t} - \hat{A}^{(0)}_{N,t-1})
   \]
   \[
   \hat{A}^{(1)}_{N,t} := \gamma \hat{A}^{(1)}_{N,t-1} + \frac{1}{\gamma} (z_{N,t} z'_{N,t-N} - \hat{A}^{(1)}_{N,t-1})
   \]
   or using the limit-memory estimation shown in (25).
5. Calculate variance by solving (23)
   \[
   \hat{\theta}_{N,t} = \hat{D}^{-1} \hat{\theta}_{N,t}
   \]
6. Return to step 3.

In conclusion, the Self-tuning Asynchronous Filter is composed of the ASYNCKF algorithm to estimate the states of process and the VAREST algorithm to estimate the noise variance and then adjust the parameter in the ASYNCKF algorithm. This is basically a improvement of the conventional methods, making it more available in practice for the asynchronous properties existing in measurements.

**IV. CASE STUDY AND SIMULATION RESULT**

This chapter will presents two experiments on the two cases mentioned in the first chapter that are applying the proposed algorithm to the wheeled robot navigation system and air quality evaluation system. It will include the mathematical model and simulation results to demonstrate the algorithm’s effectiveness.

**A. Wheeled Robot Navigation System**

The autonomous navigation in outdoor scenario using the speed/acceleration data detected by odometer or inertial measurement unit will deteriorate over time, so people often use a GPS to correct its navigation. A GPS-aided navigation system is composed of three parts: the space segment, the control segment and the user segment as illustrated in Fig. 3. The wheeled robot is mounted with a GPS receiver to determine its navigation data by measure the arrival time of signals by these satellites then calculate the three-dimensional position, speed and time. However, the GPS has low updating frequency, namely, it can not correct the output of autonomous navigation system in real-time. Another problem is the variation of noises caused by weather condition, signal block or some other disturbance.

The vehicle’s driving on roads can be approximately treated as the motion on the two-dimensional plane, which can be decomposed to the eastward motion and northward motion. Defining \( T \) as the sampling time interval, the eastward motion is described by eastward position \( x_{E,t} \) and eastward speed \( \dot{x}_{E,t} \) at time \( Tt \). Similarly, the northward motion is described by eastward position \( x_{N,t} \) and eastward speed \( \dot{x}_{N,t} \) at time \( Tt \). Based on the Newton’s laws of motion, the eastward motion and northward motion are modelled as

\[
\begin{align*}
x_{E,t+1} &= x_{E,t} + T\dot{x}_{E,t} + 0.5T^2(u_{E,t} + w_{E,t}) \\
\dot{x}_{E,t+1} &= \dot{x}_{E,t} + T(u_{E,t} + w_{E,t}) \\
x_{N,t+1} &= x_{N,t} + T\dot{x}_{N,t} + 0.5T^2(u_{N,t} + w_{N,t}) \\
\dot{x}_{N,t+1} &= \dot{x}_{N,t} + T(u_{N,t} + w_{N,t})
\end{align*}
\]

where \( u_{E,t} \) and \( u_{N,t} \) represent for eastward and northward accelerations decided by the driver respectively, \( w_{E,t} \) and \( w_{N,t} \) for random accelerations caused by the variation of wind direction. They can be modelled as Gaussian white noises with varying variance. The updating rate of GPS is usually slower than that of \( u_{E,t} \) and \( u_{N,t} \) thus we have the measurement equation as

\[
\begin{align*}
y_{E,N,t} &= x_{E,N,t} + y_{E,N,t} \\
y_{N,N,t} &= x_{N,N,t} + y_{N,N,t}
\end{align*}
\]

where \( y_{E,N,t} \) and \( y_{N,N,t} \) are the GPS-derived eastward position and northward position respectively. The noise \( v_{E,N,t} \) and \( v_{N,N,t} \) may change with time for the influence brought by amount of observable satellites or interference signal. Similarly, they can be modelled as Gaussian white noises with varying variance.

Introducing the state variable \( \mathbf{x}_t = [x_{E,t}, \dot{x}_{E,t}, x_{N,t}, \dot{x}_{N,t}]^T \), \( \mathbf{y}_t = [y_{E,N,t}, y_{N,N,t}]^T \), \( \mathbf{u}_t = [u_{E,t}, u_{N,t}]^T \), \( \mathbf{w}_t = [w_{E,t}, w_{N,t}]^T \)
Fig. 2. Simulation positioning results under $N = 5$, $N = 30$ and $N = 50$.

and $v_{Nt} = [v_{E,Nt}, v_{N,Nt}]'$, the equations from (26a) to (26d), (27a) and (27b) can be rewritten in the form of (1a), (1b), (9a) and (9b) where

$$F = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} 1 & NT & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & NT \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G = \Gamma = \begin{bmatrix} 0.5T^2 & 0 \\ T & 0 \\ 0 & 0.5T^2 \\ 0 & T \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} \sigma_E^2 & 0 \\ 0 & \sigma_N^2 \end{bmatrix}, \quad R = \begin{bmatrix} \varsigma_E^2 & 0 \\ 0 & \varsigma_N^2 \end{bmatrix}$$

Before the simulation some parameters should be determined. They are: sampling time interval $T = 0.1$, noise variance $\sigma_E^2 = \sigma_N^2 = 49$ for the first half period and $\sigma_E^2 = 900$ for the second half period and $\varsigma_E^2 = 400$ for the whole test. The results are shown in Fig. 2 where the black curves represent for real tracks and red for estimations. The upper three sub-figures is produced by the simulation under $N = 5$, $N = 30$ and $N = 50$ without using VAREXT algorithm. The lower three sub-figures is also produced by the simulation under $N = 5$, $N = 30$ and $N = 50$ but using VAREXT algorithm to adjust the coefficients of $\text{SYNCKF}$. The comparison of the upper three sub-figures shows that the navigation quality will decrease with $N$ gets large. It is found that the main errors occur on the area between 2000 and 4000 along the eastward axis in that the noise variance changes sharply at that period. What is more, the divergence is noticeable especially in Subfig. (e) for the autonomous navigation performance worse and worse without the GPS correction for a long time. However, thanks to the correction derived by GPS the deterioration will not continue. The lower three sub-figures are the corresponding simulation with the aid of VAREXT algorithm, respectively. The better performances are obvious and the errors produced by sudden change of noise variances are eliminated for the coefficients of coefficients of $\text{SYNCKF}$ are tuned adaptively.

Zooming in the Subfig (f) of Fig. 2 yield Fig. 5 where the blue circles represent for the measurements, red curves for estimation and black curves for real tracks. It is found the
Fig. 4. Simulation results for $\alpha_t$ at $t = 0, N, 2N, \ldots$ and the estimation error variance under different $N$.

data derived by autonomous navigation without correction will diverge to infinite. But the GPS will correct the estimation in time not waiting the errors to get larger. Although the performance of autonomous navigation will deteriorate over time, it performs quite well in a short period, which has complemented the shortage of GPS. Hence the wheel robot navigation system is usually integrated with the two methods by fusing their outputs, to achieve a relatively accurate estimation in real time.

B. Air Quality Evaluation System

Based on the existing dispersion model such as STREET and OSPM [23], [24], we will present a discrete linear parameterised box-type dispersion model which is basically a parameterised semi-empirical model making use of a priori assumptions about the flow and dispersion conditions. Illustrating in Fig. 6, the street can be treated as a closed box with its top exchanging air with the high-level atmosphere for the very deepness of the street canyon. The inner atmosphere is relatively stable and the pollutant concentration will decrease with a fixed proportion $k$ which is determined by the street structure. Due to the outmigration of high-pollution factories and the development in energy industry, the traffic emission has become the dominant factor of street air pollution especially in traffic-dense area. So it is assumed that the air pollution is arose almost solely by traffic emission $u$. The high-level is relatively stable such that its pollutant concentration is symbolized by a constant $\pi$. Hence, the atmospheric dispersion system can be described by an discrete linear model as

$$x_{t+1} - \pi = k(x_t - \pi) + u_t + w_t$$  \hspace{1cm} (28)

where $x_t$ represents for street (the inner box) pollutant concentration, $\pi$ for the average high-level (the external box) pollutant concentration, $u_t$ for the new concentration by traffic emission, $w_t$ for Gaussian white noise, $k \in (0, 1)$ for the dispersion coefficient and $t$ for discrete time. The equation indicates the changing principle of inner-external concentration difference over time. Considering the uncertainty existing in the model, a Gaussian white noise $w_t$ is introduced to improve its precision.

It is mentioned that the monitoring car can be used to correct the estimation not in real-time, rearranging (28) and modeling the mobile-sensing method yield the two stochastic difference equations as

$$x_{t+1} = kx_t + \chi + u_t + w_t$$ \hspace{1cm} (29a)

$$y_{Nt} = x_{Nt} + v_{Nt}$$ \hspace{1cm} (29b)

where $\chi$ equals $(1 - k)\pi$, $y_t$ for the outputs of monitoring car, $w_t \sim N(0, q)$ and $v_t \sim N(0, r)$ for two independent Gaussian
white noises. The constant $\chi$ can be seen as the time-invariant part of $u_t$, such that this model is consistent with (1a) and (1b).

Setting the parameters as dispersion coefficient $k = 0.8$, noise variance $q = 49$ for the first half period and $q = 109$ for the second half period and $r = 250$ for the whole test. The similar test will not be conducted again compared with that in Test 1. However this is a one-dimension system such that all the variables are scalar, which will facilitate us to observe the variation of estimation error variance under different $N$. It may give us a better understanding to the influence of accuracy by asynchroynism.

The results are shown in Fig. 4. The first test results shown in (a) shows the asymptotical stability which the weight $\alpha_t$ is set to be 0 initially and converges to 0.6917 and 0.4761. It can be found that the filter weights more to the measurement with $Q$ getting larger. Tests results from (b) to (f) shows the estimation error variance changing curve over time, with which the measurement are based on different sampling frequency. The curve (b) shows the estimation error variance without the correction of measurements, which stay steady at 257.9 and 889.5. Although the results are not diverging for the stability of the system own, the results can not be accepted because the variances are extremely large. The curves from (c) to (e) is based on setting $N$ as 48 12 and 3, with the error bound [126.9, 257.9], [116.4, 208.6], [93.85, 125] for the first half period and [195.1, 229.4.9], [183.5, 443.3], [155, 294.5] for the second half period. Specially noted that the $k$ is a positive number smaller than 1 which makes $K \to 0$ at $N \to \infty$. This could explain why the upper bounds at the situation $N = 48$ is same to the situation without measurements, that is mainly due to that the too long sampling time. The filter seems to reduce the estimation error variance while it is getting large so the upper bound can be significantly reduced before the variance reaches steady-state. With the sampling time of measurements getting smaller, the bound is reduced at the same time. When the measurements are synchronous where $N = 1$. thus resulting a better performance with estimation error variance equaling 77.8 and 131 for different periods as (f) shown.

This has presented a novel solution to the air pollutant concentration estimation in the atmospheric dispersion system. Compared with the former test, the system matrix $F$ equalling $k$ is stable that is all the eigenvalues are in the unit-circle. This enable the system is stable, or to say, the estimation error variance will not go to infinite if the synchronization interval is too long. The results shown in Fig. 4 have sufficiently demonstrated the effectiveness, so the other simulations will not be covered again.

V. CONCLUSION

In this paper, the filtering problem for linear Gaussian system is considered. We have proposed a self-tuning asynchronous filter subject to measurement-lossy situation where the measurements are available at equal sampling intervals, namely, the process and measurement equations are asynchronous at every time but synchronous every several times. This filter is basically an improved Kalman filter. What is more, considering the variation of noise statistics, a regular called noise variance estimator is introduced to adjust filter’s coefficients adaptively. This filter can be used in wheeled robot navigation system and air quality evaluation system and the case studies show that the stability of the system itself can prevent the divergence for a long period and the error of state estimation can be reduced sharply at the moment the measurements are available.

REFERENCES


